Paley-Wiener-Schwartz type theorem for the wavelet transform

GF2020
Ghent, Belgium
August 31–September 4, 2020

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Wavelet Transform

By dilation and translation of the basic function ψ, the wavelet $\psi_{b,a}(t)$ is defined by [5, p. 63]:

$$\psi_{b,a}(t) := |a|^{-\rho}\psi\left(\frac{t-b}{a}\right), \quad t \in \mathbb{R},\ b \in \mathbb{R}, a \in \mathbb{R}_0 = \mathbb{R}\backslash\{0\}, \rho > 0. \tag{1}$$

If $\rho = \frac{1}{2}$, then the mapping $\psi \to \psi_{b,a}$ is a unitary operator from $L^2(\mathbb{R})$ onto itself.

The wavelet transform $W(b,a)$ of f with respect to the wavelet $\psi_{b,a}(t)$ is defined by

$$W(b,a) := \int_R f(t)\overline{\psi_{b,a}(t)}dt, \tag{2}$$

provided the integral exists. If $\rho = \frac{1}{2}$ and $\psi \in L^2(\mathbb{R})$, then the wavelet transform maps each L^2-function f on $\mathbb{R}$ to a function W on $\mathbb{R} \times \mathbb{R}_0$. From Eq. (2) it follows that

$$W(b,a) = (f * \theta_{a,0})(b), \tag{3}$$

where $\theta(x) := \overline{\psi(-x)}$.

If $f \in L^p(\mathbb{R})$ and $\psi \in L^q(\mathbb{R})$ then by [2, p. 122],

$$f * \theta_{a,0}(b) \in L^r(\mathbb{R}), \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Now, applying Fourier transform:

$$\hat{f}(\omega) := \mathcal{F}(f)(\omega) = \int_{-\infty}^{\infty} f(x) e^{-ix\omega} dx, \tag{4}$$

to (3) and using convolution property, we get

$$W(b,a) = \frac{1}{2\pi} |a|^{-\rho} \int_{\mathbb{R}} e^{ib\omega} \hat{f}(\omega) \overline{\hat{\psi}(a\omega)} d\omega. \tag{5}$$

Moreover, if $f \in L^2(\mathbb{R})$ and $\psi \in L^2(\mathbb{R})$ satisfies the following admissibility condition:

$$C_\psi := \int_{\mathbb{R}} \frac{|\hat{\psi}(w)|^2}{|w|} dw < \infty, \tag{6}$$

then the following inversion formula for the wavelet transform with $\rho = \frac{1}{2}$, holds:

$$\frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}_0} \frac{1}{\sqrt{|a|}} W(b,a) \psi\left(\frac{x-b}{a}\right) \frac{dbda}{a^2} = f(x). \tag{7}$$

The existent applications of wavelet methods in mathematical analysis are rich. The requirements of modern mathematics, mathematical physics and engineering, need to incorporate ideas from wavelet analysis to the distribution theory.

Testing function space $G_{\alpha,\beta}(\mathbb{R})$ and its dual

Let us recall the definition of the space $G_{\alpha,\beta}(\mathbb{R})$ from [5, pp. 48-49]. Assume that a positive and continuous function $\zeta_{\alpha,\beta}(t)$ on $\mathbb{R}$ is given by

$$\zeta_{\alpha,\beta}(t) = \begin{cases} e^{\alpha t} & 0 \leq t < \infty \\ \\ e^{\beta t} & -\infty < t < 0, \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$.

Then $G_{\alpha,\beta}(\mathbb{R})$ denotes the space of all complex-valued smooth functions $\psi(t)$ on $-\infty < t < \infty$ such that for each $k = 0, 1, 2, \ldots,$

$\gamma_k(\psi) = \sup_{t\in R}|\zeta_{\alpha,\beta}(t) D^k \psi(t)| < \infty$, where $D^k = (\frac{d}{dt})^k$, $k = 0, 1, 2, \ldots.$

$G_{\alpha,\beta}$ is a vector space. The topology over $G_{\alpha,\beta}$ is generated by the sequence of seminorms $\{\gamma_k\}_{k=0}^{\infty}$ [5]. A sequence $\{\psi_\nu\}_{\nu=1}^{\infty}$ is a Cauchy sequence in $G_{\alpha,\beta}$ if for each non-negative integer k, $\gamma_k(\psi_\mu - \psi_\nu) \to 0$ as $\mu, \nu \to \infty$ independently of each other. The space $G_{\alpha,\beta}$ is a sequentially complete space and therefore it is a complete countably multinormed space and so a Fréchet space. $\mathcal{D}$ is the space of smooth functions on $\mathbb{R}$ having compact support. The topology of $\mathcal{D}$ is that which makes its dual the space $\mathcal{D}'$ of Schwartz distributions on $\mathbb{R}$. Since $\mathcal{D} \subset G_{\alpha,\beta}$ and the topology of $\mathcal{D}$ is stronger than that induced on $\mathcal{D}$ by $G_{\alpha,\beta}$, it follows that the restriction of any $f \in G'_{\alpha,\beta}$ to $\mathcal{D}$ is in $\mathcal{D}'$. For details, see ([5]).

Lemma

If $\psi \in G_{\alpha,\beta}$, *then* $\psi(\frac{t-b}{a}) \in G_{\alpha,\beta}$ *for* $\alpha \leq 0$ *and* $\beta \geq 0$ *when* $|a| \geq 1$ *and* $\psi(\frac{t-b}{a}) \in G_{\alpha,\beta}$ *for* $\alpha \geq 0$ *and* $\beta \leq 0$ *when* $0 < |a| < 1$.

Distributional Wavelet Transform

We assume $\psi \in G_{\alpha,\beta}(\mathbb{R})$ is the basic function generating the wavelet $\psi_{b,a}(t)$ given in Eq. (1). Since function $\psi\left(\frac{t-b}{a}\right)$ belongs to $G_{\alpha,\beta}$ for fixed b and $a \neq 0$ as a function of t under condition of Lemma 1, for $f \in G'_{\alpha,\beta}$ the wavelet transform $W(b,a)$ of f is defined by

$$W(b,a) = \frac{1}{\sqrt{|a|}}\left\langle f(t),\ \psi\left(\frac{t-b}{a}\right)\right\rangle, \quad a \in \mathbb{R}_0,\ b \in \mathbb{R}. \tag{8}$$

For convenience, in what follow we shall deal with

$$\widetilde{W}(b,a) = \left\langle f(t),\ \psi\left(\frac{t+b}{a}\right)\right\rangle, \quad a \in \mathbb{R}_0,\ b \in \mathbb{R}, \tag{9}$$

instead of $W(b,a)$.

Theorem

Let $f \in G'_{\alpha,\beta}$, $\psi \in G_{\alpha,\beta}$ and $\widetilde{W}(b,a)$ be defined by Eq. (9). Then $\widetilde{W}(b,a)$ is smooth and

$$D_b^k \widetilde{W}(b,a) = \left\langle f(t),\ D_b^k \psi\left(\frac{t+b}{a}\right)\right\rangle, \quad k = 1, 2, 3...$$

and

$$D_a^k \widetilde{W}(b,a) = \left\langle f(t),\ D_a^k \psi\left(\frac{t+b}{a}\right)\right\rangle, \quad k = 1, 2, 3...$$

Remark

Using change of variables and following the above technique differentiability of $W(b,a)$ is also be proved.

Theorem

For real b and $a \in \mathbb{R}_0$ let $W(b,a)$ be defined as in Eq. (8), then under conditions of Lemma 1,

$$W(b,a) = O\left(\frac{1}{|a|^{k+\frac{1}{2}}}\right), \quad |a| \to 0, \quad \textit{for some } k \in \mathbb{N}.$$

Inversion of the Distributional Wavelet Transform

In order to derive inversion formula for the distributional wavelet transform, we construct a structure formula for the distribution $f \in G'_{\alpha,\beta}$ for $\alpha, \beta > 0$ [4, pp. 272-274]. If $f \in G'_{\alpha,\beta}$ and $\phi \in G_{\alpha,\beta}$, then by boundedness property of distributions, there exists a $C > 0$ and a non-negative integer m satisfying

$$|\langle f, \phi \rangle| \leq C \max_{0 \leq k \leq m} \sup_{t \in R_+} \left| e^{\alpha t} D_t^k \phi(t) \right|, \quad \forall t \geq 0, \tag{10}$$

and

$$|\langle f, \phi \rangle| \leq C \max_{0 \leq k \leq m} \sup_{t \in R_-} \left| e^{\beta t} D_t^k \phi(t) \right|, \quad \forall t < 0. \tag{11}$$

Let m be the least possible value of the non-negative integer. Then

$$
\begin{aligned}
|\langle f, \phi\rangle| &\leq C \max_{0\leq k\leq m} \sup_{t\in R_+} \left| \int_{-\infty}^{t} \left| \frac{d}{dt} \left[e^{\alpha t} D_t^k \phi(t) \right] dt \right| \right| \\
&\leq C \max_{0\leq k\leq m} \sup_{t\in R_+} \left| \int_{-\infty}^{t} \left| e^{\alpha t} D_t^{k+1} \phi(t) + \alpha e^{\alpha t} D_t^k \phi(t) \right| dt \right| \\
&\leq C \sum_{k=0}^{m} \int_{-\infty}^{\infty} \left| e^{\alpha t} D_t^{k+1} \phi(t) + \alpha e^{\alpha t} D_t^k \phi(t) \right| dt \\
&\leq C' \sum_{k=0}^{m} \left\| e^{\alpha t} (D_t + t) D_t^k \phi(t) \right\|_2 .
\end{aligned}
$$

Now using the Hahn-Banach theorem and the Riesz representation theorem we get g_k belonging to the space $L^2(R)$ satisfying

$$|\langle f, \phi \rangle| = \sum_{k=0}^{m} \left\langle g_k(t), e^{\alpha t}(D_t + t) D_t^k \phi(t) \right\rangle .$$

Therefore our structure formula is (for $t \geq 0$)

$$f = \sum_{k=0}^{m} (-1)^{k+1} D_t^k (D_t - t) \left\{ e^{\alpha t} g_k(t) \right\}, \tag{12}$$

similarly for $t < 0$

$$f = \sum_{k=0}^{m} (-1)^{k+1} D_t^k (D_t - t) \left\{ e^{\beta t} g_k(t) \right\}. \tag{13}$$

We now establish the inversion formula for the distributional wavelet transform using Eq. (7).

Theorem

Assume that the wavelet transform $W(b,a)$ of $f \in G'_{\alpha,\beta}$ is given by Eq. (8). Then

$$\lim_{\substack{N\to\infty \\ R\to\infty}} \left\langle \frac{1}{C_\psi} \int_{-R}^{R} \int_{-N}^{N} W(b,a)\psi_{b,a}(x) \frac{dbda}{a^2},\ \phi(x) \right\rangle = \langle f, \phi \rangle, \quad (14)$$

for each $\phi \in \mathcal{D}$, $a \in \mathbb{R}_0$ and $b \in \mathbb{R}$, where $\psi_{b,a}(x)$ is defined by (1) with $\rho = 1$.

Example

Let us consider the Mexican hat wavelet,defined by

$$\psi(t) = (1 - t^2)\exp(-t^2/2) = -\frac{d^2}{dt^2}\exp(-t^2/2),$$

and its Fourier transform is defined by

$$\hat{\psi}(w) = \sqrt{2\pi}w^2\exp(-w^2/2).$$

It is a C^∞- function and well localized in time and frequency domains. The k^{th} derivative of Mexican hat wavelet given by

$$D^k\psi(t) = \sum_{r=0}^{k}\binom{k}{r}D^{(r)}(1 - t^2)D^{(k-r)}\exp(-t^2/2).$$

Example

Using property of Hermite polynomial [3] we write the last expression as

$$D^k\psi(t) = \sum_{r=0}^{k} \binom{k}{r} D^{(r)}(1-t^2)\exp(-t^2/2)H_{k-r}(t)$$

$$= k(1-t^2)\exp(-t^2/2)H_k(t) + \binom{k}{r}(-2t)\exp(-t^2/2)H_{k-1}(t),$$

where $H_k(t)$ denotes the Hermite polynomial. Therefore, $\psi(t) \in G_{\alpha,\beta}(\mathbb{R})$ and the Mexican hat wavelet transform of $f \in G'_{\alpha,\beta}$ is then defined by Eq. (8).

Paley-Wiener-Schwartz type theorem

Now, we discuss on extension of the wavelet transform on distribution spaces of compact support and develop the Paley-Wiener-Schwartz type theorem for the wavelet transform. Paley-Wiener-Schwartz type theorem for the wavelet transform as double Fourier transform is also established.

Wavelet transform in Fourier space

In this section we assume that wavelets are such that their Fourier transforms are of compact support. To deal with such wavelets we suppose that $\psi \in \mathscr{S}(\mathbb{R})$, then $\psi_{b,a} \in \mathscr{S}(\mathbb{R})$ for fixed $a \in \mathbb{R}_0, b \in \mathbb{R}$. We extend the wavelet transform in Fourier space defined by

$$W(b,a) = \frac{|a|^{1/2}}{2\pi} \int_{\mathbb{R}} e^{ib\omega} \hat{f}(\omega) \overline{\hat{\psi}(a\omega)} d\omega. \tag{15}$$

Assume that $\hat{f}(\omega) \in \mathcal{D}'(\mathbb{R})$ and $\overline{\hat{\psi}(\omega)} \in \mathscr{S}(\mathbb{R})$ is of compact support, then $\overline{\hat{\psi}(a\omega)} \hat{f}(\omega) \in \mathscr{E}'(\mathbb{R})$.

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Generalized wavelet transform

Now, we define generalized wavelet transform of $f \in \mathcal{Z}'(\mathbb{R})$ as generalized inverse Fourier transform of $\hat{f}(\cdot)\overline{\hat{\psi}(a\cdot)}$:

$$W(b,a) = \frac{|a|^{1/2}}{2\pi}\left\langle \hat{f}(\omega),\ \overline{\hat{\psi}(a\omega)}e^{ib\omega}\right\rangle = \frac{|a|^{1/2}}{2\pi}\left\langle \hat{f}(\omega)\overline{\hat{\psi}(a\omega)},\ e^{ib\omega}\right\rangle$$
$$= \frac{1}{2\pi}\left\langle \frac{1}{|a|^{1/2}}\hat{f}(u/a)\overline{\hat{\psi}(u)},\ e^{ibu/a}\right\rangle = \left\langle \frac{1}{|a|^{1/2}}g_a(u),\ e^{ibu/a}\right\rangle. \tag{16}$$

where $g_a(u) = \hat{f}(u/a)\overline{\hat{\psi}(u)} \in \mathscr{E}'(\mathbb{R}),\ a \in \mathbb{R}_0$.

Assume that $\operatorname{supp}\hat{\psi}(u) = [-r,r],\ r > 0$. Then $\operatorname{supp} g_a(u) = [-r,r],\ r > 0$.

Theorem

(1a) The wavelet transform of distribution g_a with compact support in $\mathbb{R}$ is an entire function $W(\zeta, a)$ of the complex variable $\zeta = b + i\eta \in \mathbb{C}$, satisfying the following property:
There are constants C and r and an integer $N \geq 0$ such that

$$|W(\zeta, a)| \leq \frac{C}{|a|^{1/2}}(1 + |\zeta/a|)^N e^{r|\eta/a|}, \ \forall \zeta \in \mathbb{C}. \tag{17}$$

(1b) Conversely, if $W(\zeta, a)$ is an entire function in $\mathbb{C} \times \mathbb{R}_0$ which satisfies (17), then $W(b, 1/a)$ is the double Fourier transform of a distribution belonging to $\mathscr{E}'(\mathbb{R}^2)$.
(2a) The wavelet transform of an infinitely differentiable function with compact support in $\mathbb{R}$ is an entire function $W(\zeta, a)$ in $\mathbb{C} \times \mathbb{R}_+$ satisfying the following property:

Theorem (cont.)

There are constant C and $r > 0$ and an integer $N \geq 0$ such that

$$|W(\zeta, a)| \leq \frac{C}{|a|^{1/2}}(1 + |\zeta/a|)^{-N} e^{r|\eta/a|}, \ \forall \zeta \in \mathbb{C}, a \in \mathbb{R}_+. \quad (18)$$

(2b) Assume that $\psi \in \mathscr{S}(\mathbb{R})$. Then every entire function on $\mathbb{C} \times \mathbb{R}_0$ satisfying (18) is the wavelet transform of a decreasing C^∞-function f such that $f * \bar{\psi}(x)$ is of compact support.

Paley-Wiener-Schwartz type theorem for the wavelet transform as double Fourier transform of distribution spaces

The following relations between wavelet transform on $\mathbb{R}$ and two dimensional Fourier transform is given in [1]:

$$F(b,a) := |a|^{-1/2} W(b,a) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{iub+iva}\, \overline{\hat{\psi}(v/u)} \frac{\hat{f}(u)}{|u|} dvdu \tag{19}$$

where $b \in \mathbb{R}, a \in \mathbb{R}_0 = \mathbb{R}\backslash\{0\}$.

Theorem

(1a) Let $W(\zeta,\tau)$, $\zeta = b + i\eta, \tau = a + i\xi$ be the extended wavelet transform:

$$W(\zeta,\tau) = \left(\frac{1}{2\pi}\right)^2 \left\langle |\tau|^{1/2}\Phi(u,v),\ e^{i\zeta u + i\tau v}\right\rangle$$

of f such that the function $\Phi(u,v) := \overline{\hat{\psi}(v/u)}\frac{\hat{f}(u)}{|u|} \in \mathscr{E}'(\mathbb{R}^2)$ with $supp(\Phi) \subset \overline{B} = \{(u,v) : u^2 + v^2 \leq r^2\}$. Then for every positive integer N there exists a positive constant C_N such that

$$|W(\zeta,\tau)| \leq C_N|\tau|^{1/2}(1 + |\zeta+\tau|)^N e^{r|Im(\zeta+\tau)|}. \tag{20}$$

Theorem (cont.)

(1b) Conversely, every entire function in $\mathbb{C}^2$ *satisfying (20) is the double Fourier transform of a distribution belonging to* $\mathscr{E}'(\mathbb{R}^2)$. *However,* $W(\zeta, a) = F^{-1}[|a|^{1/2}\hat{f}(\cdot)\overline{\hat{\psi}(a\cdot)}](\zeta)\ \forall a \in \mathbb{R}_0$, *which satisfies (20) is the wavelet transform of a distribution* $\Phi(u, v) \in \mathscr{E}'(\mathbb{R}^2)$. *(2a) Assume that* f *and* ψ *such that* $\Phi(u, v) \in C_c^\infty(\mathbb{R}^2)$. *Then the wavelet transform of* f *is an entire function* $W(\zeta, a)$ *in* $\mathbb{C}^2$ *satisfying the following property:*
There are constant C *and* $r > 0$ *and an integer* $N \geq 0$ *such that*

$$|W(\zeta, \tau)| \leq C_N(1 + |\zeta + \tau|)^{-N} e^{r|Im(\zeta+\tau)|}. \tag{21}$$

(2b) Conversely, every entire function on $\mathbb{C}^2$ *satisfying (21) is the double Fourier transform of a* C^∞*-function with compact support in* $\mathbb{R}^2$.

Generalization of Paley-Wiener-Schwartz type theor

Theorem

The wavelet transform of an entire function g_a satisfying the condition

$$|g_a(z)| \leq \frac{C}{\sqrt{|a|}}\left(1+\left|\frac{z}{a}\right|\right)^M e^{r\left|\frac{y}{a}\right|} \; for \; all \; z \in \mathbb{C}, \tag{22}$$

considered as a functional on $\mathscr{E}'(\mathbb{R})$, has the form

$$W(b,a) = \sum_{k=1}^{m} P_k\left(\frac{\partial}{\partial b}\right) F_k(b). \tag{23}$$

Conversely, the wavelet transform of a functional of the form (23) is an entire function satisfying (22).

In order to give an example, we extend the Mexican hat wavelet transform in the Fourier space as follows:

We consider Mexican hat wavelet transform defined by Pathak *et al.* [2, p.470] as

$$(Wf)(b,a) = (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{-\infty}^{\infty} f(t) D_t^2 k(b-t, a^2) dt, \qquad (24)$$

where

$$k(b-t, a^2) = k_{a^2}(b-t) = \frac{1}{(2\pi)^{\frac{1}{2}} a} e^{-\frac{(b-t)^2}{2a^2}}, \; b \in \mathbb{C}, a \in \mathbb{R}_+.$$

$$
\begin{aligned}
(Wf)(b,a) &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{-\infty}^{\infty} f(t) D_t^2 k_{a^2}(b-t)\,dt \\
&= -(2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{\mathbb{R}} \hat{f^{(2)}}(w) \hat{k_{a^2}}(w) e^{ibw}\,dw. \qquad (25)
\end{aligned}
$$

Taking

$$
\begin{aligned}
\hat{k_{a^2}}(w) &= \frac{1}{(2\pi)^{\frac{1}{2}} a} \int_{\mathbb{R}} e^{-\frac{t^2}{(2a^2)}} e^{-iwt}\,dt \\
&= e^{-\frac{a^2 w^2}{2}}.
\end{aligned}
$$

Therefore, (25) becomes

$$\begin{aligned}(Wf)(b,a) &= -(2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{\mathbb{R}} \hat{f^{(2)}}(w) e^{-\frac{a^2 w^2}{2}} e^{ibw} dw \\ &= -(2\pi)^{\frac{1}{2}} a^{\frac{3}{2}} \left\langle \frac{1}{a} g_a^{(2)}(u), e^{\frac{ibu}{a}} \right\rangle .\end{aligned}$$

The above relation extends the Mexican hat wavelet transform as generalized inverse Fourier transform of a function $g_a^{(2)}(u) = \hat{f^{(2)}}\left(\frac{u}{a}\right) e^{-\frac{u^2}{2}}$, $a \in \mathbb{R}_+$. Thus, using this relation we define the following example for the generalized Paley-Wiener- Schwartz type theorem.

Example

Consider an entire function $g_a^{(2)}(u) \in \mathscr{E}'(\mathbb{R})$ satisfying

$$\left|g_a^{(2)}(u)\right| \leq \frac{C'}{a^{\frac{3}{2}}}\left(1+\left|\frac{u}{a}\right|\right)^{M'} e^{r\left|\frac{y}{a}\right|} e^{-\frac{y^2}{2}}.$$

Then by using Theorem 11, its Mexican hat wavelet transform can be expressed as:

$$(Wf)(b,a) = \sum_{k=1}^{m} P_k\left(\frac{\partial}{\partial b}\right) F_k^{(2)}(b),$$

where

$$|F_k^{(2)}(b,a)| \leq C' a^{\frac{3}{2}} \int_{-\infty}^{\infty} \frac{|b|^l \left(1+\left|\frac{u}{a}\right|\right)^{M'} e^{r\left|\frac{y}{a}\right|} e^{-\frac{y^2}{2}} e^{\frac{-yb}{a}}}{\left(\left(\frac{u}{a}\right)^2 - \left(\frac{y}{a}\right)^2 + Cb^{2\beta} + 1\right)^p} du.$$

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Thank You for Your Attention !